International Journal of Professional Development
Vol.12, No.1, Jan-JUNE 2023 ISSN: 2277-517X (Print), 2279-0659 (Online)
Impact Factor: 3.984(IFSIJ)

Common Fixed Point Theorems of Contractive type Mappings

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Abstract
Fixed point theory is a fundamental area in mathematics with significant applications in various fields, including optimization, differential equations, and computer science. Among the numerous branches of this theory, common fixed point theorems for contractive-type mappings hold particular importance. These theorems provide conditions under which two or more mappings on a metric or generalized metric space share a common fixed point.
Contractive-type mappings, characterized by their ability to bring points closer together under certain constraints, serve as a cornerstone for establishing these theorems. This abstract explores key developments in common fixed point theorems, focusing on mappings such as Banach contractions, Kannan mappings, and Reich-type contractions. It also examines the extension of these results to broader settings, including metric-like spaces, partially ordered spaces, and fuzzy metric spaces, enhancing their applicability to real-world problems.
The study underscores the significance of contractive conditions in guaranteeing the existence and uniqueness of common fixed points. Furthermore, it highlights recent advancements in this area, driven by innovations in metric space generalizations and the incorporation of hybrid mappings. These results not only enrich the theoretical framework of fixed point theory but also extend its applicability to interdisciplinary domains, paving the way for future research.
Keywords: Common Fixed Point, Theorems, Mappings

Introduction

Fixed point theory is a central area in mathematical analysis and has become a cornerstone for solving problems across numerous disciplines, such as engineering, computer science, physics, economics, and biology. The concept of a fixed point, which refers to a point that remains invariant under a given mapping, provides a powerful framework for addressing equations and systems where equilibrium or stability plays a critical role. The significance of this theory lies in its universality and adaptability, offering solutions to problems ranging from nonlinear differential equations to optimization and game theory.

A mapping T is said to have a fixed point if there exists a point xxx in the domain of T such that
T(x)=xT(x) = xT(x)=x.

While this concept appears simple, the conditions under which fixed points exist are highly nuanced and require deep mathematical insights. The development of fixed point theorems, which establish criteria for the existence and uniqueness of fixed points, has been one of the most remarkable achievements in the field of mathematics. Among these, Banach's Fixed Point Theorem, commonly referred to as the Contraction Mapping Principle, is perhaps the most celebrated. It asserts that any contraction mapping on a complete metric space has a unique fixed point, and it provides a constructive method for finding that point.

Beyond individual fixed points, the study of common fixed points, where two or more mappings share a single fixed point, has attracted significant attention. Common fixed point theorems are critical in extending the applicability of fixed point theory to systems involving multiple mappings. These results are essential for understanding coupled systems, iterative algorithms, and equilibrium analysis in dynamic environments. Contractive-type mappings, which generalize the notion of contraction, have proven to be particularly effective in deriving common fixed point theorems.

The importance of contractive-type mappings lies in their ability to bridge theory with practical applications. These mappings extend the classical contraction condition by introducing various generalized criteria, such as weak contractions, Kannan mappings, and Reich-type contractions. These generalized notions enable fixed point results in spaces and contexts where traditional conditions may fail. For example, in spaces that are not complete or in settings involving non-linear or non-Euclidean structures, contractive-type mappings allow the existence of fixed points to be established.

In recent years, the scope of fixed point theory has expanded beyond metric spaces to encompass generalized spaces, such as metric-like spaces, fuzzy metric spaces, and partial metric spaces. These generalizations reflect the increasing complexity of modern problems that arise in fields such as data analysis, machine learning, and network theory. The versatility of fixed point theorems in these new frameworks underscores their foundational role in advancing both theoretical and applied mathematics.

One significant motivation for the study of fixed points is their practical relevance. In optimization, fixed point methods are used to find solutions to variational problems and equilibrium systems. In computer science, they are instrumental in algorithms for data clustering, image reconstruction, and machine learning. Physics relies on fixed point theory to analyze stability in dynamical systems, while economics uses it to determine market equilibria and Nash equilibria in game theory. The universal applicability of fixed point results demonstrates their profound influence across disciplines.

At the heart of this study is the exploration of common fixed point theorems for contractive-type mappings. These theorems not only generalize classical results but also open doors to new methods and applications. By analyzing mappings that satisfy relaxed contractive conditions, such as cyclic contractions or hybrid mappings, researchers have made significant strides in solving problems that are more complex and multidimensional.

This introduction also highlights the methodological significance of fixed point theory. Iterative techniques, often associated with fixed point results, provide computationally efficient tools for solving equations and systems of equations. These methods, rooted in the constructive nature of fixed point theorems, are particularly useful in applied settings, where closed-form solutions are often unattainable.

Despite its successes, fixed point theory faces challenges and opportunities for growth. The development of fixed point results in generalized spaces, such as those with incomplete structures or non-linear metrics, remains an area of active research. Furthermore, the integration of fixed point methods with modern computational tools, such as artificial intelligence and big data analytics, presents exciting possibilities for addressing real-world problems.

In conclusion, fixed point theory and its extensions, particularly through contractive-type mappings, represent a vibrant and impactful area of study. The study of common fixed points is a natural progression that expands the applicability and utility of this theory. By addressing the interplay between theoretical advancements and practical applications, this field continues to contribute to the broader understanding of mathematical structures and their relevance to modern science and technology. This paper aims to explore the foundations, advancements, and applications of common fixed point theorems for contractive-type mappings, offering insights into their significance and potential for future development.

Preliminaries provide foundational concepts, definitions, and mathematical structures necessary for understanding fixed point theorems and their applications.

Definitions

Metric Space:

A metric space (X,d)(X, d)(X,d) is a set XXX equipped with a distance function d:X×X→Rd : X \times X \to \mathbb{R}d:X×X→R satisfying:

- d(x,y)≥0d(x, y) \geq 0d(x,y)≥0 and d(x,y)=0 ⟺ x=yd(x, y) = 0 \iff x = yd(x,y)=0⟺x=y (non-negativity and identity of indiscernibles).
- d(x,y)=d(y,x)d(x, y) = d(y, x)d(x,y)=d(y,x) (symmetry).

- d(x,z)≤d(x,y)+d(y,z)d(x, z) \leq d(x, y) + d(y, z)d(x,z)≤d(x,y)+d(y,z) (triangle inequality).

Fixed Point:

Let T:X→XT: X \to XT:X→X be a mapping. A point x∈Xx \in Xx∈X is a fixed point of TTT if T(x)=xT(x) = xT(x)=x.

Contraction Mapping:

A mapping T:X→XT: X \to XT:X→X is a contraction if there exists a constant k∈[0,1)k \in [0, 1)k∈[0,1) such that d(T(x),T(y))≤k⋅d(x,y)d(T(x), T(y)) \leq k \cdot d(x, y)d(T(x),T(y))≤k⋅d(x,y) for all x,y∈Xx, y \in Xx,y∈X.

Common Fixed Point:

Two mappings T_1,T_2:X→XT_2, T_2: X \to XT_1,T_2 :X→X have a common fixed point if there exists x∈Xx \in Xx∈X such that T_1(x)=T_2(x)=xT_1(x) = T_2(x) = xT_1(x)=T_2(x)=x.

Propositions

Proposition 1 (Banach Contraction Mapping Theorem):

If (X,d)(X, d)(X,d) is a complete metric space and T:X→XT: X \to XT:X→X is a contraction mapping, then T has a unique fixed point x∗∈Xx^* \in Xx∗∈X. Furthermore, the sequence defined by xn+1=T(xn)x_{n+1} = T(x_n)xn+1 =T(xn), starting from any x0∈Xx_0 \in Xx0∈X, converges to x∗x^*x∗.

Proposition 2 (Common Fixed Point Theorem for Commuting Mappings):

Let (X,d)(X, d)(X,d) be a complete metric space, and let T1,T2:X→XT_1, T_2: X \to XT_1,T_2:X→X be two commuting contraction mappings (i.e., T(T_2(x))=T_2(T_1(x))T_1(T_2(x)) = T_2 (T_1(x))T_1(T_2 (x))=T_2(T1(x)) for all x∈Xx \in Xx∈X). Then T_1 and T_2 have a unique common fixed point.

Proposition 3 (Kannan's Fixed Point Theorem):

Let (X,d)(X, d)(X,d) be a complete metric space, and let T:X→XT: X \to XT:X→X be a mapping such that for all x,y∈Xx, y \in Xx,y∈X:

d(T(x),T(y))≤12[d(x,T(x))+d(y,T(y))].d(T(x),

Then T has a unique fixed point.

Proposition 4 (Reich's Contraction Mapping Theorem):

Let (X,d)(X, d)(X,d) be a complete metric space, and let T:X→XT: X \to XT:X→X satisfy:

d(T(x),T(y))≤αd(x,y)+β[d(x,T(x))+d(y,T(y))]d(T(x),

for all x,y∈Xx, y \in Xx,y∈X, where α,β≥0\alpha, \beta \geq 0α,β≥0 and α+2β<1\alpha + 2\beta < 1α+2β<1. Then T has a unique fixed point.

These definitions and propositions form the foundation for understanding fixed point theory and its applications to contractive-type mappings.

Let me know if you'd like additional details or more advanced results!

Partially Ordered Set (Poset):

A set X is a partially ordered set if it is equipped with a binary relation ≤\leq≤ satisfying:

- Reflexivity: x≤xx \leq xx≤x for all x∈Xx \in Xx∈X.
- Antisymmetry: If x≤yx \leq yx≤y and y≤xy \leq xy≤x, then x=yx = yx=y.
- Transitivity: If x≤yx \leq yx≤y and y≤zy \leq zy≤z, then x≤zx \leq zx≤z.

Complete Metric Space:

A metric space (X,d)(X, d)(X,d) is complete if every Cauchy sequence in X converges to a point in X.

- A sequence {xn}\{x_n\}{xn} is Cauchy if for every ϵ>0\epsilon > 0ϵ>0, there exists N∈NN \in \mathbb{N}N∈N such that d(xm,xn)<ϵd(x_m, x_n) < \epsilond(xm ,xn)<ϵ for all m,n≥Nm, n \geq Nm,n≥N.

Self-Mapping:

A mapping T:X→XT: X \to XT:X→X is called a self-mapping if the domain and codomain of TTT are the same set X.

Cyclic Mapping:

Let X_1,X_2,...,X_k, be non-empty subsets of a metric space X. A mapping T:$\bigcup^k$=1Xi→⋃i=1kXiT: \bigcup_{i=1}^k X_i \to \bigcup_{i=1}^k X_iT:⋃i=1kXi→⋃i=1kXi is called cyclic if T(Xi)⊆Xi+1T(X_i)

Weak Contraction:

A mapping T:X→XT: X \to XT:X→X is a weak contraction if there exists a constant k∈[0,1)k \in [0, 1)k∈[0,1) and a non-decreasing function ϕ:R+→R+\phi: \mathbb{R}^+ \to \mathbb{R}^+ϕ:R+→R+ with ϕ(0)=0\phi(0) = 0ϕ(0)=0 such that for all x,y∈Xx, y \in Xx,y∈X:

d(T(x),T(y))≤k⋅d(x,y)+ϕ(d(x,y)).

Multivalued Mapping:

A mapping T:X→2XT: X \to 2^XT:X→2X (the power set of X) is called a multivalued mapping if T(x)T(x)T(x) is a subset of X for every x∈Xx \in Xx∈X.

A fixed point of T is x∈Xx \in Xx∈X such that x∈T(x)x \in T(x)x∈T(x).

Convex Metric Space:

A metric space (X,d)(X, d)(X,d) is convex if for any two points x,y∈Xx, y \in Xx,y∈X and t∈[0,1]t \in [0, 1]t∈[0,1], there exists a point z∈Xz \in Xz∈X such that d(z,x)=t·d(x,y)d(z, x)

Fuzzy Metric Space:

A fuzzy metric space (X,M,∗)(X, M, *)(X,M,∗) consists of a set XXX, a fuzzy set M:X×X×R+→[0,1]M: X \times X \times \mathbb{R}^+ \to [0, 1]M:X×X×R+→[0,1], and a t-norm ∗, satisfying specific axioms. This framework generalizes metric spaces to handle uncertainty.

Hybrid Mapping:

A mapping T:X→XT: X \to XT:X→X is hybrid if it combines properties of multiple contraction conditions. For example, it may satisfy a Kannan-type contraction on part of the domain and a Banach-type contraction on another.

Asymptotic Contraction:

A mapping T:X→XT: X \to XT:X→X is an asymptotic contraction if there exists a sequence {kn}\{k_n\}{kn} with kn→0k_n \to 0kn→0 as n→∞n \to \infty n→∞ such that:

d(Tn(x),Tn(y))≤kn·d(x,y)∀x,y∈X, n∈N

Non-Expansive Mapping:

A mapping T:X→Xis non-expansive X is non-expansive if for all x,y∈Xx, y \in Xx,y∈X:

d(T(x),T(y))≤d(x,y)

Iterated Function System (IFS):

An iterated function system is a collection of mappings {Ti:X→X}i=1n\{T_i: X \to X\}_{i=1}^n{Ti:X→X}i=1n used to construct fractals through iterative application. Fixed points of IFS play a key role in fractal generation.

Lemma 1: Basic Fixed Point Lemma for Contraction Mapping

Let (X,d)(X, d)(X,d) be a complete metric space, and let T:X→XT: X \to XT:X→X be a contraction mapping. Then:

1. T has a unique fixed point.
2. For any x0∈Xx_0 \in Xx0∈X, the sequence {xn}\{x_n\}{xn} defined by xn+1=T(xn)x_{n+1} = T(x_n)xn+1 =T(xn) converges to the fixed point.

Lemma 2: Monotone Sequence Lemma in Ordered Metric Spaces

Let (X,≤,d)(X, \leq, d)(X,≤,d) be a partially ordered complete metric space. Suppose T:X→XT: X \to XT:X→X is an order-preserving mapping (i.e., x≤y ⟹ T(x)≤T(y)x \leq y \implies T(x) \leq T(y)x≤y⟹T(x)≤T(y)) and there exists a lower bound x0∈Xx_0 \in Xx0∈X such that x0≤T(x0)x_0 \leq T(x_0)x0≤T(x0). Then:

- The iterative sequence {xn}\{x_n\}{xn} defined by xn+1=T(xn)x_{n+1} = T(x_n)xn+1=T(xn) is monotone increasing.
- If X is complete, {xn}\{x_n\}{xn} converges to a fixed point of T.

Lemma 3: Contraction Condition Lemma

If a mapping T:X→XT: X \to XT:X→X satisfies d(T(x),T(y))≤kd(x,y)d(T(x), T(y)) \leq kd(x, y)d(T(x),T(y))≤kd(x,y) for all x,y∈Xx, y \in Xx,y∈X and k∈[0,1)k \in [0, 1)k∈[0,1), then TTT is continuous.

Lemma 4: Fixed Point Lemma for Cyclic Mappings

Let (X,d)(X, d)(X,d) be a complete metric space, and let T:X→XT: X \to XT:X→X be a cyclic mapping with subsets X1,X2,...,Xk⊆XX_1, X_2, ..., X_k \subseteq XX1,X2,...,Xk⊆X such that T(Xi)⊆Xi+1T(X_i) \subseteq X_{i+1}T(Xi)⊆Xi+1 (Xk+1=X1X_{k+1} = X_1Xk+1=X1). If T satisfies a contraction condition:

d(T(x),T(y))≤k·d(x,y)∀x,y∈⋃i=1kXi, k∈[0,1),

then T has a unique fixed point in ⋃i=1kXi\bigcup_{i=1}^k X_i⋃i=1kXi.

Lemma 5: Krasnoselskii's Lemma

Let X be a Banach space, and let A,B:X→XA, B: X \to XA,B:X→X be mappings such that:

1. A is a contraction.
2. B is continuous and compact.
3. Then, the operator T=A+BT = A + BT=A+B has at least one fixed point.

Lemma 6: Banach's Lemma (Iterative Approximation)

For any contraction mapping T:X→XT: X \to XT:X→X in a complete metric space (X,d)(X, d)(X,d), the iterative sequence {xn}\{x_n\}{xn} defined by xn+1=T(xn)x_{n+1} = T(x_n)xn+1 =T(xn) converges to the fixed point x∗x^*x∗, and the convergence rate satisfies:

d(xn,x∗)≤knd(x0,x1),d(x_n, x^*) \leq k^n d(x_0, x_1),d(xn,x∗)≤knd(x0,x1),

where kkk is the contraction constant.

Lemma 7: Common Fixed Point Lemma for Commuting Mappings

Let T_1,T_2:X→XT_1, T_2: X \to XT_1,T_2:X→X be two commuting self-mappings on a complete metric space (X,d)(X, d)(X,d) (i.e., $T_{(}1T_2$(x))=T_2(T_1(x))T_1(T_2(x)) = T_2(T_1(x))T_1(T_2(x))=$T_{2(}T_{1(}$x)) for all x∈Xx \in Xx∈X). If both T_1 and T_2 are contractions, then they have a unique common fixed point.

Lemma 8: Fixed Point Lemma for Convex Spaces

Let X be a convex metric space, and let T:X→XT: X \to XT:X→X be a non-expansive mapping (d(T(x),T(y))≤d(x,y)d(T(x), T(y)) \leq d(x, y)d(T(x),T(y))≤d(x,y) for all x,y∈Xx, y \in Xx,y∈X). If T is asymptotically regular (i.e., lim⁡n→∞d(Tn(x),Tn+1(x))=0\lim_{n \to \infty} d(T^n(x), T^{n+1}(x)) = 0limn→∞ d(Tn(x),Tn+1(x))=0 for all x∈Xx \in Xx∈X), then T has at least one fixed point.

These lemmas establish crucial foundational results and provide tools for proving more advanced theorems in fixed point theory. Let me know if you'd like detailed explanations of any lemma or its proof!

Theorem 1: Banach Fixed Point Theorem (Contraction Mapping Theorem)

Let (X,d)(X, d)(X,d) be a complete metric space and T:X→XT: X \to XT:X→X a contraction mapping, i.e., there exists k∈[0,1)k \in [0, 1)k∈[0,1) such that:

d(T(x),T(y))≤k⋅d(x,y)∀x,y∈X.d(T(x), T

Then:

1. T has a unique fixed point x∗∈Xx^* \in Xx∗∈X.
2. For any initial point x0∈Xx_0 \in Xx0∈X, the sequence defined by xn+1=T(xn)x_{n+1} = T(x_n)xn+1 =T(xn) converges to x∗x^*x∗.
3. The rate of convergence is geometric with factor kkk.

Theorem 2: Schauder Fixed Point Theorem

Let C be a non-empty, closed, bounded, and convex subset of a Banach space X, and let T:C→CT: C \to CT:C→C be a continuous mapping. Then T has at least one fixed point in C.

Theorem 3: Brouwer Fixed Point Theorem

Let X be a non-empty, compact, and convex subset of a Euclidean space Rn\mathbb{R}^nRn, and let T:X→XT: X \to XT:X→X be a continuous mapping. Then T has at least one fixed point.

Theorem 4: Kakutani Fixed Point Theorem

Let CCC be a non-empty, compact, and convex subset of a finite-dimensional vector space, and let T:C→2CT: C \to 2^CT:C→2C (a multivalued mapping) satisfy:

1. T(x)T(x)T(x) is non-empty, closed, and convex for all x∈Cx \in Cx∈C.
2. T has a closed graph.
3. Then T has at least one fixed point.

Theorem 5: Knaster-Tarski Fixed Point Theorem

Let (X,≤)(X, \leq)(X,≤) be a complete lattice, and let T:X→XT: X \to XT:X→X be a monotone function (order-preserving). Then T has at least one fixed point in X, and the set of fixed points forms a complete lattice.

Theorem 6: Boyd-Wong Fixed Point Theorem

Let (X,d)(X, d)(X,d) be a complete metric space and T:X→XT: X \to XT:X→X satisfy the following generalized contraction condition:

d(T(x),T(y))≤ϕ(d(x,y))∀x,y∈X,d(T(x),

where ϕ:[0,∞)→[0,∞)\phi: [0, \infty) \to [0, \infty)ϕ:[0,∞)→[0,∞) is a function such that ϕ(t)<t\phi(t) < tϕ(t)<t for all t>0t > 0t>0.

Then T has a unique fixed point in X.

Theorem 7: Edelstein Fixed Point Theorem

Let XXX be a compact metric space, and let T:X→XT: X \to XT:X→X satisfy the condition:

d(T(x),T(y))<d(x,y)∀x,y∈X, x≠y.

Then T has a unique fixed point in X.

Theorem 8: Caristi Fixed Point Theorem

Let (X,d)(X, d)(X,d) be a complete metric space, and let φ:X→[0,∞)\varphi: X \to [0, \infty)φ:X→[0,∞) be a lower semi-continuous function. If T:X→XT: X \to XT:X→X satisfies:

d(x,T(x))≤φ(x)−φ(T(x))∀x∈X,d(x,

then T has a fixed point in X.

Theorem 9: The Fixed Point Theorem for Multivalued Contractions

Let (X,d)(X, d)(X,d) be a complete metric space, and let T:X→2XT: X \to 2^XT:X→2X be a multivalued mapping such that T(x)T(x)T(x) is non-empty, closed, and satisfies:

H(T(x),T(y))≤k⋅d(x,y)∀x,y∈X, k∈[0,1),

where H is the Hausdorff metric. Then T has a fixed point.

Theorem 10: Fixed Point Theorem in Partially Ordered Metric Spaces

Let (X,≤,d)(X, \leq, d)(X,≤,d) be a partially ordered complete metric space. Suppose T:X→XT: X \to XT:X→X is a monotone mapping (order-preserving) and there exists x0∈Xx_0 \in Xx0∈X such that x0≤T(x0)x_0 \leq T(x_0)x0≤T(x0). If TTT satisfies a contraction condition:

d(T(x),T(y))≤k⋅d(x,y)∀x,y∈X, k∈[0,1),

then T has a fixed point in X.

These main results demonstrate the versatility of fixed point theory and its applicability to various mathematical frameworks, including metric spaces, Banach spaces, and lattice structures. Let me know if you'd like more information or specific examples related to these theorems!

Conclusion

Fixed point theory serves as a cornerstone in mathematical analysis and its applications, providing a robust framework for understanding the existence and uniqueness of solutions to a wide array of problems across disciplines. From Banach's and Brouwer's foundational theorems to modern advancements involving multivalued and partially ordered spaces, the theory has evolved to address increasingly complex scenarios. Its relevance extends beyond pure mathematics, influencing fields such as economics, computer science, physics, and engineering, where iterative processes and equilibrium states are studied.

The unifying nature of fixed point theory bridges diverse areas, offering insights into nonlinear analysis, optimization, and dynamical systems. It forms the backbone of algorithms used in solving differential equations, game theory, and modeling ecological systems. The interplay between theoretical generalizations, such as Schauder's and Kakutani's fixed point theorems, and practical applications highlights the adaptability and significance of this field.

As research continues to expand into areas such as quantum computing, machine learning, and interdisciplinary studies, fixed point theory remains integral in addressing emerging challenges. Its potential for fostering collaboration between mathematics and applied sciences underscores its enduring importance. In conclusion, fixed point theory is not just a mathematical abstraction but a critical tool for solving real-world problems and advancing knowledge across multiple domains.

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